UNIT–III

10

CHAPTER — LAPLACE TRANSFORM

10.1 INTRODUCTION

Laplace transforms help in solving the differential equations with boundary values without finding the general solution and the values of the arbitrary constants.

10.2 LAPLACE TRANSFORM

Definition. Let $f(t)$ be function defined for all positive values of t, then

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

provided the integral exists, is called the **Laplace Transform** of $f(t)$. It is denoted as

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

10.3 IMPORTANT FORMULAE

	$f(t)$	$L f(t) = F(s)$
1.	1	$\frac{1}{s}$
2.	t	$\frac{1}{s^2}$
3.	t^n	$\frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, 3...$
4.	e^{at}	$\frac{1}{s-a}$
5.	e^{-at}	$\frac{1}{s+a}$
6.	$\sin at$	$\frac{a}{s^2+a^2}$ $(s > 0)$
7.	$\cos at$	$\frac{s}{s^2+a^2}$ $(s > 0)$
8.	$\sinh at$	$\frac{a}{s^2-a^2}$ $(s > 0)$
9.	$\cosh at$	$\frac{s}{s^2-a^2}$ $(s^2 > a^2)$

Proof.

1. $\boxed{L(1)=\frac{1}{s}}$

Proof. $L(1) = \int_0^{\infty} 1\cdot e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$

$$= -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^{\infty} = -\frac{1}{s}[0-1] = \frac{1}{s}.$$

Hence $L(1) = \frac{1}{s}$ **Proved.**

2. $\boxed{L(t)=\frac{1}{s^2}}$

Proof. $L(t) = \int_0^{\infty} t\cdot e^{-st}\,dt = \left[t\,\frac{e^{-st}}{-s} - \frac{e^{-st}}{(s-a)^2}\right]_0^{\infty}$

$$=\left[0-0+0+\frac{1}{s^2}\right] = \frac{1}{s^2}$$ **Proved.**

3. $\boxed{L(t^n)=\frac{n!}{s^{n+1}}}$ where n and s are positive.

Proof. $L(t^n) = \int_0^{\infty} e^{-st}t^n\,dt$

Putting $st = x$ or $t=\frac{x}{s}$ or $dt = \frac{dx}{s}$

Thus, we have $L(t^n) = \int_0^{\infty} e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s}$

$\Rightarrow$ $L(t^n) = \frac{1}{s^{n+1}}\int_0^{\infty} e^{-x}.x^n\,dx$

$\Rightarrow$ $L(t^n) = \frac{\overline{|n+1}}{s^{n+1}}$

$\Rightarrow$ $L(t^n) = \frac{n!}{s^{n+1}}$ $\left[\overline{|n+1} = \int_0^{\infty} e^{-x}x^n\,dx \text{ and } \overline{|n+1} = n!\right]$ **Proved.**

4. $\boxed{L(e^{at})=\frac{1}{s-a}}$ **where $s > a$**

Proof. $L(e^{at}) = \int_0^{\infty} e^{-st}.e^{at}\,dt = \int_0^{\infty} e^{-st+at}.dt$

$$= \int_0^\infty e^{(-s+a)t}.dt = \int_0^\infty e^{-(s-a)t}.dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$$ Proved.

5. $\boxed{L\left(e^{-at}\right) = \frac{1}{s+a}}$

Proof. $L(e^{-at}) = \int_0^\infty e^{-st}.e^{-at}dt = \int_0^\infty e^{-st-at}.dt$

$$= \int_0^\infty e^{-(s+a)t}.dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= -\frac{1}{s+a}\left[\frac{1}{e^{(s+a)t}}\right]_0^\infty = -\frac{1}{s+a}(0-1) = \frac{1}{s+a}$$ Proved.

6. $\boxed{L(\sin at) = \frac{a}{s^2+a^2}}$

Proof. $L(\sin at) = L\left[\frac{e^{iat}-e^{-iat}}{2i}\right]$ $\left[\because \sin at = \frac{e^{iat}-e^{-iat}}{2i}\right]$

$$= \frac{1}{2i}\left[L\left(e^{iat}-e^{-iat}\right)\right] = \frac{1}{2i}\left[L\left(e^{iat}\right)-L\left(e^{-iat}\right)\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia}-\frac{1}{s+ia}\right] = \frac{1}{2i}\frac{s+ia-s+ia}{s^2+a^2}$$

$$= \frac{1}{2i}\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$ **Proved.**

7. $\boxed{L(\cos at) = \frac{s}{s^2+a^2}}$

Proof. $L(\cos at) = L\left(\frac{e^{iat}+e^{-iat}}{2}\right)$ $\left[\because \cos at = \frac{e^{iat}+e^{-iat}}{2}\right]$

$$= \frac{1}{2}[L(e^{iat}+e^{-iat})] = \frac{1}{2}[L(e^{iat})+L(e^{-iat})]$$

$$= \frac{1}{2}\left[\frac{1}{s-ia}+\frac{1}{s+ia}\right] = \frac{1}{2}\frac{s+ia+s-ia}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$ Proved.

8. L(sinh at)

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9. L(cosh at)

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8. $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

Proof. $L(\sinh\ at) = L\left[\frac{1}{2}\left(e^{at} - e^{-at}\right)\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

Proved.

9. $L(\cosh at) = \dfrac{s}{s^2 - a^2}$

Proof. $L(\cosh at) = L\left[\frac{e^{at} + e^{-at}}{2}\right]$ $\left(\therefore \cosh at = \frac{e^{at} + e^{-at}}{2}\right)$

$$= \frac{1}{2}L\left(e^{at}\right) + \frac{1}{2}L\left(e^{-at}\right)$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \qquad \left[L\left(e^{at}\right) = \frac{1}{s-a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

Proved.

Example 1. *Find the Laplace transform of f(t) defined as*

$$f(t) = \begin{cases} \frac{t}{k}, & \text{when } 0 < t < k \\ 1, & \text{when } t > k \end{cases}$$

Solution. $L[f(t)] = \int_0^k \frac{t}{k} e^{-st} dt + \int_k^\infty 1.e^{-st} dt = \frac{1}{k}\left[\left(t\frac{e^{-st}}{-s}\right)_0^k - \int_0^k \frac{e^{-st}}{-s} dt\right] + \left[\frac{e^{-st}}{-s}\right]_k^\infty$

$$= \frac{1}{k}\left[\frac{ke^{-ks}}{-s} - \left(\frac{e^{-st}}{s^2}\right)_0^k\right] + \frac{e^{-ks}}{s} = \frac{1}{k}\left[\frac{ke^{-ks}}{-s} - \frac{e^{-sk}}{s^2} + \frac{1}{s^2}\right] + \frac{e^{-ks}}{s}$$

$$= -\frac{e^{-sk}}{s} - \frac{1}{k}\frac{e^{-ks}}{s^2} + \frac{1}{k}\frac{1}{s^2} + \frac{e^{-ks}}{s} = \frac{1}{ks^2}[-e^{-ks} + 1]$$

Ans.

Example 2. *Find the Laplace transform of the function* $f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$.

(U.P., II Semester, 2009)

Solution. The given function is periodic with period 3.

$$L[f(t)] = \int_1^3 f(t)\,e^{-st} dt = \left[\int_1^2 f(t)\,e^{-st} dt + \int_2^3 f(t)\,e^{-st} dt\right]$$

$$= \left[\int_1^2 (t-1)\, e^{-st} dt + \int_2^3 (3-t)\, e^{-st} dt\right]$$

$$= \left[\left\{(t-1)\frac{e^{-st}}{-s} - \frac{e^{-st}}{(-s)^2}\right\}_1^2 + \left\{(3-t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{(-s)^2}\right\}_2^3\right]$$

$$= \left[\left\{\frac{e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2}\right\} + \left\{\frac{e^{-3s}}{s^2} - \frac{e^{-2s}}{-s} - \frac{e^{-2s}}{s^2}\right\}\right]$$

$$= \left[-\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right]$$

$$= \left[\frac{1}{s^2}(-e^{-2s} + e^{-s} + e^{-3s} - e^{-2s})\right] = \frac{1}{s^2}[e^{-s} - 2e^{-2s} + e^{-3s}]$$ Ans.

Example 3. *Find the Laplace transform of* $F(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & 2 \le t < \infty \end{cases}$ (*Q. Bank U.P. 2001*)

Solution. Here, we have $F(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & 2 \le t < \infty \end{cases}$

$$L[F(t)] = \int_0^\infty e^{-st} \cdot F(t) dt = \int_0^1 e^{-st} dt + \int_1^2 t\, e^{-st} dt + \int_2^\infty t^2 e^{-st} dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^1 + \left(t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right)_1^2 + \left(t^2 \frac{e^{-st}}{-s}\right)_2^\infty - \int_2^\infty 2t \cdot \frac{e^{-st}}{-s} dt$$

$$= \left(\frac{1-e^{-s}}{s}\right) + \left(\frac{-2}{s}e^{-2s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right) + \frac{4}{s}e^{-2s} + \frac{2}{s}\int_2^\infty t\, e^{-st} dt$$

$$= \frac{1}{s} + \frac{2}{s}e^{-2s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2}{s}\left[\left(t\frac{e^{-st}}{-s}\right)_2^\infty - \int_2^\infty 1 \cdot \frac{e^{-st}}{-s} dt\right]$$

$$= \frac{1}{s} + \frac{2}{s}e^{-2s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2}{s}\left[\frac{2}{s}e^{-2s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_2^\infty\right]$$

$$= \frac{1}{s} + \frac{2}{s}e^{-2s} + \frac{e^{-s}}{s^2} + \frac{3}{s^2}e^{-2s} + \frac{2}{s^3}e^{-2s}.$$ Ans.

Example 4. *Find the Laplace transform of*

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$$ (*U.P. II Semester, June 2007*)

Solution. $L[f(t)] = \int_0^\infty e^{-st} f(t) dt = \int_0^2 t^2 e^{-st} dt + \int_2^3 (t-1) e^{-st} dt + \int_3^\infty 7e^{-st} dt$

$$\left[\int I\, II = I\, II_1 - I'\, II_{11} + I''\, II_{111} - \ldots\right]$$

$$=\left[t^2\left(\frac{e^{-st}}{(-s)}\right)-2t\frac{e^{-st}}{(-s)^2}+2\frac{e^{-st}}{(-s)^3}\right]_0^2+\left[(t-1)\left(\frac{e^{-st}}{(-s)}\right)-\frac{e^{-st}}{(-s)^2}\right]_2^3+7\left[\frac{e^{-st}}{-s}\right]_3^{\infty}$$

$$=\left[-4\left(\frac{e^{-2s}}{s}\right)-4\left(\frac{e^{-2s}}{s^2}\right)-2\left(\frac{e^{-2s}}{s^3}\right)+\frac{2}{s^3}\right]+\left[2\left(\frac{e^{-3s}}{-s}\right)-\frac{e^{-3s}}{s^2}+\frac{e^{-2s}}{s}+\frac{e^{-2s}}{s^2}\right]+7\left(0+\frac{e^{-3s}}{s}\right)$$

$$=\frac{2}{s^3}+e^{-2s}\left[-\frac{4}{s}-\frac{4}{s^2}-\frac{2}{s^3}\right]+e^{-3s}\left[-\frac{2}{s}-\frac{1}{s^2}\right]+e^{-2s}\left[\frac{1}{s}+\frac{1}{s^2}\right]+e^{-3s}\left[\frac{7}{s}\right]$$

$$=\frac{2}{s^3}+e^{-2s}\left[-\frac{4}{s}-\frac{4}{s^2}-\frac{2}{s^3}+\frac{1}{s}+\frac{1}{s^2}\right]+e^{-3s}\left[-\frac{2}{s}-\frac{1}{s^2}+\frac{7}{s}\right]$$

$$=\frac{2}{s^3}+e^{-2s}\left[-\frac{3}{s}-\frac{3}{s^2}-\frac{2}{s^3}\right]+e^{-3s}\left[\frac{5}{s}-\frac{1}{s^2}\right]=\frac{2}{s^3}-\frac{e^{-2s}}{s^3}\left(2+3s+3s^2\right)+\frac{e^{-3s}}{s^2}(5s-1)$$ **Ans.**

Example 5. *Find the Laplace transform of (1 + sin 2t).*

Solution. Laplace transform of (1 + sin 2t)

$$=\int_0^{\infty}e^{-st}(1+\sin 2t)\,dt=\int_0^{\infty}e^{-st}\left(1+\frac{e^{2it}-e^{-2it}}{2i}\right)dt$$

$$=\frac{1}{2i}\int_0^{\infty}[2ie^{-st}+e^{(-s+2i)t}-e^{(-s-2i)t}]\,dt=\frac{1}{2i}\left[\frac{2ie^{-st}}{-s}+\frac{e^{(-s+2i)t}}{-s+2i}-\frac{e^{(-s-2i)t}}{-s-2i}\right]_0^{\infty}$$

$$=\frac{1}{2i}\left[\left(0+\frac{2i}{s}\right)+\frac{1}{-s+2i}(0-1)-\frac{1}{-s-2i}(0-1)\right]$$

$$=\frac{1}{2i}\left[\frac{2i}{s}+\frac{1}{s-2i}-\frac{1}{s+2i}\right]=\frac{1}{2}\left[\frac{2}{s}+\frac{4}{s^2+4}\right]=\frac{1}{s}+\frac{2}{s^2+4}$$ **Ans.**

Alternate Method

$$L(1+\sin 2t)=L(1)+L\sin 2t=\frac{1}{s}+\frac{2}{s^2+4}$$ **Ans.**

10.4 PROPERTIES OF LAPLACE TRANSFORM

10.4 (1) $L[af_1(t)+bf_2(t)]=a\,L[f_1(t)]+b\,L[f_2(t)]$

Proof. $L[af_1(t)+bf_2(t)]=\int_0^{\infty}e^{-st}[af_1(t)+bf_2(t)]\,dt$

$$=a\int_0^{\infty}e^{-st}f_1(t)\,dt+b\int_0^{\infty}e^{-st}f_2(t)\,dt$$

$$=a\,L[f_1(t)]+b\,L[f_2(t)]$$ **Proved.**

10.4 (2) Change of Scale Property

If $L\{f(t)\}=F(s)$ then $\boxed{L\{f(at)\}=\frac{1}{a}F\left(\frac{s}{a}\right)}$

$$=\left[t^2\left(\frac{e^{-st}}{(-s)}\right)-2t\frac{e^{-st}}{(-s)^2}+2\frac{e^{-st}}{(-s)^3}\right]_0^2+\left[(t-1)\left(\frac{e^{-st}}{(-s)}\right)-\frac{e^{-st}}{(-s)^2}\right]_2^3+7\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$=\left[-4\left(\frac{e^{-2s}}{s}\right)-4\left(\frac{e^{-2s}}{s^2}\right)-2\left(\frac{e^{-2s}}{s^3}\right)+\frac{2}{s^3}\right]+\left[2\left(\frac{e^{-3s}}{-s}\right)-\frac{e^{-3s}}{s^2}+\frac{e^{-2s}}{s}+\frac{e^{-2s}}{s^2}\right]+7\left(0+\frac{e^{-3s}}{s}\right)$$

$$=\frac{2}{s^3}+e^{-2s}\left[-\frac{4}{s}-\frac{4}{s^2}-\frac{2}{s^3}\right]+e^{-3s}\left[-\frac{2}{s}-\frac{1}{s^2}\right]+e^{-2s}\left[\frac{1}{s}+\frac{1}{s^2}\right]+e^{-3s}\left[\frac{7}{s}\right]$$

$$=\frac{2}{s^3}+e^{-2s}\left[-\frac{4}{s}-\frac{4}{s^2}-\frac{2}{s^3}+\frac{1}{s}+\frac{1}{s^2}\right]+e^{-3s}\left[-\frac{2}{s}-\frac{1}{s^2}+\frac{7}{s}\right]$$

$$=\frac{2}{s^3}+e^{-2s}\left[-\frac{3}{s}-\frac{3}{s^2}-\frac{2}{s^3}\right]+e^{-3s}\left[\frac{5}{s}-\frac{1}{s^2}\right]=\frac{2}{s^3}-\frac{e^{-2s}}{s^3}\left(2+3s+3s^2\right)+\frac{e^{-3s}}{s^2}(5s-1)$$ **Ans.**

Example 5. *Find the Laplace transform of (1 + sin 2t).*

Solution. Laplace transform of $(1+\sin 2t)$

$$=\int_0^\infty e^{-st}(1+\sin 2t)\,dt=\int_0^\infty e^{-st}\left(1+\frac{e^{2it}-e^{-2it}}{2i}\right)dt$$

$$=\frac{1}{2i}\int_0^\infty[2ie^{-st}+e^{(-s+2i)t}-e^{(-s-2i)t}]dt=\frac{1}{2i}\left[\frac{2ie^{-st}}{-s}+\frac{e^{(-s+2i)t}}{-s+2i}-\frac{e^{(-s-2i)t}}{-s-2i}\right]_0^\infty$$

$$=\frac{1}{2i}\left[\left(0+\frac{2i}{s}\right)+\frac{1}{-s+2i}(0-1)-\frac{1}{-s-2i}(0-1)\right]$$

$$=\frac{1}{2i}\left[\frac{2i}{s}+\frac{1}{s-2i}-\frac{1}{s+2i}\right]=\frac{1}{2}\left[\frac{2}{s}+\frac{4}{s^2+4}\right]=\frac{1}{s}+\frac{2}{s^2+4}$$ **Ans.**

Alternate Method

$$L(1+\sin 2t)=L(1)+L\sin 2t=\frac{1}{s}+\frac{2}{s^2+4}$$ **Ans.**

10.4 PROPERTIES OF LAPLACE TRANSFORM

10.4 (1) $L[af_1(t)+bf_2(t)]=a\,L[f_1(t)]+b\,L[f_2(t)]$

Proof. $L[af_1(t)+bf_2(t)]=\int_0^\infty e^{-st}[af_1(t)+bf_2(t)]\,dt$

$$=a\int_0^\infty e^{-st}f_1(t)\,dt+b\int_0^\infty e^{-st}f_2(t)\,dt$$

$$=a\,L[f_1(t)]+b\,L[f_2(t)]$$ **Proved.**

10.4 (2) Change of Scale Property

If $L\{f(t)\}=F(s)$ then $\boxed{L\{f(at)\}=\frac{1}{a}F\left(\frac{s}{a}\right)}$

Proof. $L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt$

$$= \int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\frac{du}{a} \qquad \left[\text{Put } at = u \Rightarrow dt = \frac{du}{a}\right]$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\,du = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)\,dt$$

$$= \frac{1}{a}\int_0^\infty e^{-St} f(t)\,dt = \frac{1}{a}L\{f(t)\} = \frac{1}{a}F(S) \qquad \left[\text{Put } S = \frac{s}{a}\right]$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

Proved.

Example 6. *If* $L(\sin t) = \frac{1}{s^2+1}$ *find* L (sin 5*t*).

Solution. $L(\sin t) = \frac{1}{s^2+1}$

$$L(\sin 5t) = \frac{1}{5}\cdot\frac{1}{\left(\frac{s}{5}\right)^2+1} = \frac{5}{s^2+25} \qquad \left(s \text{ should be replaced by } \frac{s}{5}\right)$$

Ans.

Example 7. *If* $L\{f(t)\} = \frac{40+5s}{s^2+6s+1}$, *then find* $L\ \{f(2t)\}$.

Solution. $L\{f(t)\} = \frac{40+5s}{s^2+6s+1}$

$$L\ \{f(2t)\} = \frac{1}{2}\cdot\frac{40+5\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2+6\left(\frac{s}{2}\right)+1} = \frac{80+5s}{s^2+12s+4}$$

Ans.

Example 8. *Find the Laplace transform of* $\cos^2 t$.

Solution. We know that $\cos 2t = 2\cos^2 t - 1$

$$\cos^2 t = \frac{1}{2}[\cos 2t + 1]$$

$$L(\cos^2 t) = L\left[\frac{1}{2}(\cos 2t+1)\right] = \frac{1}{2}[L(\cos 2t)+L(1)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(2)^2}+\frac{1}{s}\right] = \frac{1}{2}\left[\frac{s}{s^2+4}+\frac{1}{s}\right]$$

Ans.

Example 9. *If* $L(\cos^2 t) = \frac{s^2+2}{s(s^2+4)}$, *find* L ($\cos^2 at$). *(U.P., II Semester, Summer 2006)*

Solution. We have, $L(\cos^2 t) = \frac{s^2+2}{s(s^2+4)}$

By change of scale property, we have

$$\ldots^2 at) = \frac{1}{a}\cdot\frac{\left(\frac{s}{a}\right)^2+2}{\frac{s}{a}\left[\left(\frac{s}{a}\right)^2+4\right]} = \frac{1}{a}\left[\frac{s^2+2a^2}{\frac{s}{a}\left(s^2+4a^2\right)}\right]$$

$$= \frac{s^2+2a^2}{s\left(s^2+4a^2\right)}$$ **Ans.**

Example 10. *If* $L\{J_0(\sqrt{t})\} = \frac{e^{-\frac{1}{4s}}}{s}$, *find* $L\{J_0(2\sqrt{t})\}$.

Solution. Here, we have

$$L\{J_0(\sqrt{t})\} = \frac{e^{-\frac{1}{4s}}}{s}$$

By change of scale property,

$$L\{J_0(\sqrt{4t})\} = \frac{1}{4}\cdot\left\{\frac{e^{-\frac{1}{4(s/4)}}}{(s/4)}\right\}$$

$$\Rightarrow \quad L\{J_0(2\sqrt{t})\} = \frac{1}{s}e^{-1/s}$$ **Ans.**

Example 11. *Find the Laplace transform of* $t^{-\frac{1}{2}}$.

Solution. We know that $L(t^n) = \frac{\lceil n+1}{s^{n+1}}$

Put $n = -\frac{1}{2}$, $L\left(t^{-1/2}\right) = \frac{\lceil -\frac{1}{2}+1}{s^{-1/2+1}} = \frac{\lceil \frac{1}{2}}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}}$, where $\lceil \frac{1}{2} = \sqrt{\pi}$ **Ans.**

Example 12. *Find the Laplace transform of 2 sin 2t cos 4t.*

Solution. We have

$$f(t) = 2\sin 2t\cos 4t$$

$$= \sin(2t+4t) + \sin(2t-4t)$$

$$= \sin 6t - \sin 2t$$

$$L f(t) = L(\sin 6t) - L(\sin 2t)$$

$$= \frac{6}{s^2+36} - \frac{2}{s^2+4}$$ **Ans.**

Example 13. *Find the Laplace transform of* $4\sin^3 t$.

Solution. We have

$$f(t) = 4\sin^3 t$$

$$= 3\sin t - \sin 3t \qquad [\sin 3t = 3\sin t - 4\sin^3 t]$$

$$L f(t) = 3L\sin t - L\sin 3t$$

$$= \frac{3}{s^2+1} - \frac{3}{s^2+9}$$ **Ans.**

Example 14. *Find the Laplace transform of 4 cosh 2t sin 4t*

Solution. We have

$$f(t) = 4\cosh 2t \sin 4t = 4\left(\frac{e^{2t}+e^{-2t}}{2}\right)\left(\frac{e^{4it}-e^{-4it}}{2i}\right)$$

$$= \frac{1}{i}\left[e^{(2+4i)t} - e^{(2-4i)t} + e^{(-2+4i)t} - e^{(-2-4i)t}\right]$$

$$L[f(t)] = -i\left[L(e^{(2+4i)t}) - L(e^{(2-4i)t}) + L(e^{(-2+4i)t}) - L(e^{(-2-4i)t})\right]$$

$$= -i\left[\frac{1}{s-2-4i} - \frac{1}{s-2+4i} + \frac{1}{s+2-4i} - \frac{1}{s+2+4i}\right]$$

$$= -i\left[\left(\frac{1}{s-2-4i} - \frac{1}{s+2+4i}\right) - \left(\frac{1}{s-2+4i} - \frac{1}{s+2-4i}\right)\right]$$

$$= -i\left[\frac{4+8i}{s^2-(2+4i)^2} - \frac{4-8i}{s^2-(2-4i)^2}\right]$$

EXERCISE 10.1

Find the Laplace transforms of the following:

1. $t + t^2 + t^3$ — **Ans.** $\frac{1}{s^2} + \frac{2}{s^3} + \frac{6}{s^4}$
2. $\sin t \cos t$ — **Ans.** $\frac{1}{s^2+4}$
3. $t^{7/2} e^{5t}$ (*M.D.U. Dec. 2009*) — **Ans.** $\frac{105\sqrt{\pi}}{16(s-5)^{9/2}}$
4. $\sin^3 2t$ — **Ans.** $\frac{48}{(s^2+4)(s^2+36)}$
5. $e^{-t}\cos^2 t$ — **Ans.** $\frac{1}{2s+2} + \frac{s+1}{2s^2+4s+10}$
6. $\sin 2t \cos 3t$ — **Ans.** $\frac{2(s^2-5)}{(s^2+1)(s^2+25)}$
7. $\sin 2t \sin 3t$ — **Ans.** $\frac{12s}{(s^2+1)(s^2+25)}$
8. $\cos at \sinh at$ — **Ans.** $\frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}\right]$
9. $\sinh^3 t$ — **Ans.** $\frac{6}{(s^2-1)(s^2-9)}$

   $\frac{3\sinh t + \sinh 3t}{4}$, $\frac{3}{4}\left(\frac{1}{s^2-1}\right) + \frac{1}{4}\left(\frac{3}{s^2-9}\right)$, $\frac{3}{4}\left[\frac{1}{s^2-1} + \frac{1}{s^2-9}\right]$

10. $\cos t \cos 2t$ — **Ans.** $\frac{s(s^2+5)}{(s^2+1)(s^2+9)}$
11. $\cosh at \sin at$ — **Ans.** $\frac{a(s^2+2a^2)}{s^4+4a^4}$

12. $f(t)=\begin{cases}\cos\left(t-\frac{2\pi}{3}\right), & t>\frac{2\pi}{3}\\ 0, & t<\frac{2\pi}{3}\end{cases}$ **Ans.** $e^{\frac{-2\pi s}{3}}\cdot\frac{s}{s^2+1}$

10.5 EXISTENCE THEOREM

According to this theorem $\int_0^\infty e^{-st} f(t)\,dt$ exists if $\int_0^\lambda e^{-st} f(t)dt$ can actually be evaluated and its limit as $\lambda\to\infty$ exists. Otherwise we may use the following theorem:

If $f(t)$ is continuous and $\lim_{t\to\infty}\left[e^{-at}f(t)\right]$ is finite, then Laplace transform of $f(t)$ *i.e.* $\int_0^\infty e^{-st}f(t)dt$ exists for $s>a$.

It should however, be kept in mind that the above foresaid conditions are sufficient but not necessary.

For example; $L\left(\frac{1}{\sqrt{t}}\right)$ exists though $\frac{1}{\sqrt{t}}$ is infinite at $t=0$. Similarly a function $f(t)$ for which $\lim_{t\to\infty}\left[e^{-at}f(t)\right]$ is finite and having a finite discontinuity will have a Laplace transform of $s>a$.

10.6 FIRST SHIFTING THEOREM.

If $L\{f(t)\}=F(s)$, then (*MTU, Jan. 2013*)

$$L[e^{at}f(t)]=F(s-a)$$

Proof. $L[e^{at}f(t)] = \int_0^\infty e^{-st}.e^{at}f(t)dt=\int_0^\infty e^{-(s-a)t}f(t)dt$

$= \int_0^\infty e^{-rt}f(t)dt$ where $r=s-a$

$= F(r)=F(s-a)$ **Proved.**

With the help of this property, we can have the following important results :

1. $L(e^{at}t^n)=\frac{n!}{(s-a)^{n+1}}$
2. $L(e^{at}\sin bt)=\frac{b}{(s-a)^2+b^2}$
3. $L(e^{at}\cos bt)=\frac{s-a}{(s-a)^2+b^2}$
4. $L(e^{at}\sinh bt)=\frac{b}{(s-a)^2-b^2}$
5. $L(e^{at}\cosh bt)=\frac{s-a}{(s-a)^2-b^2}$

10.7 LAPLACE TRANSFORM OF THE DERIVATIVE OF f(t)

$L[f'(t)]=sL[f(t)]-f(0)$ where $L[f(t)]=F(s)$.

Proof. $L[f'(t)]=\int_0^\infty e^{-st}f'(t)dt$